Abstract—The starting point of any derivation is a suitable representation of the given model. Hypercomplex numbers sometimes provide a more compact representation and more insight into a problem’s structure than the reals or the complex numbers. Hence, efficient filters are needed for hypercomplex numbers as well. As there is a large zoo of different hypercomplex numbers obeying different algebras it is cumbersome to do derivations for each of them individually.

Hence, our contribution is to show how to abstract the concept of hypercomplex numbers and their algebras. We give an insight into the questions how the algebra works in general. Furthermore, we propose two extensions of the concept of widely linear filters for hypercomplex numbers.

The first widely linear filter abstraction presumes several properties of the algebra, but can be computed directly in the respective hypercomplex domain. The second abstract solution of widely linear filters does all calculations in the real domain. The latter imposes much less restrictions on the algebra than the first one which leads to a more generic type of widely linear filters.

I. INTRODUCTION

Complex numbers give physicists, engineers and computer scientists a tool for describing signals and systems in a convenient and compact way. However, some problems in these scientific disciplines are better described in different ways as they may have more than one component or even obey different rules.

Hypercomplex numbers generalize the idea of complex numbers by incorporating different amounts of imaginary units and different rules of calculation (algebras). For instance, one of the most well known ideas concerning hypercomplex numbers is the use of quaternions for dual-polarized antenna systems as described in [12]. As a result of these and other findings in the past years a zoo of applications of hypercomplex numbers [4], [5] has been developed and is still subject to scientific studies.

Nowadays, hypercomplex algebras are not only utilized in wireless communications systems to exploit polarization effects or to increase direction of arrival estimation as shown in [7]. Image processing and vector image processing have already [8] been introduced. Moreover, hypercomplex numbers have been successfully applied in acoustics [11], wind profile forecasting [10], epileptic EEG classification [13] and other scientific disciplines.

Under ideal conditions complex numbers show no correlation with their complex conjugate. Such data is said to be circular or proper. However, this assumption is often violated and leads to degradation in filter performance. The problem of improper data is combated by employing both the data and its conjugate [3] using widely linear filters.

This idea has recently been extended to the quaternion-valued case [6]. To that end, multiple quaternion conjugates were defined to describe the complete statistics of the data. We followed this path of abstraction in our work. As a result, we show how to apply this idea to nearly any hypercomplex algebra. At the end, we will propose two abstractions of widely linear filters. The first type of abstract filters can be employed by using the respective algebra directly, whereas the second type makes use of the real notation of any hypercomplex number. The latter one has the advantage of imposing much less constraints on the algebra itself.

The remainder of the paper is organized as follows: In section II we introduce the notation used throughout the paper and give a short review on the general concept of hypercomplex numbers. Based on this, the widely linear filter data model is explained in section III. Section IV shows how to derive the filter coefficients. Section V concludes the paper.

II. HYPERCOMPLEX NUMBERS - REVIEW AND NOTATION

This section provides a short overview about the mathematical concept of hypercomplex numbers and the associated algebras. A hypercomplex representation is given as well as a vectorized real representation. Afterwards the rules of conjugation and multiplication are given as generic functions.

A. General Notation

Throughout the whole paper we use the following notation: Vectors are indicated either by lower-case letters in bold type set or by underlined expressions. Capital letters in bold type refer to matrices. The operator $^T$ is used to indicate the transpose of a matrix. In addition to the conventional matrix multiplication we will use the $\otimes$ operator for the Kronecker product. The vec $(\cdot)$ operator is used to return a vector which has all the columns of the given matrix stacked one below the other. Furthermore, the $\text{tr} (\cdot)$ function computes the trace of a matrix, i.e., the sum of its diagonal elements. A diagonal
matrix is obtained from a vector via the diag (·) operator. The expectation operator is denoted by \( \mathbb{E} \{ \cdot \} \).

**B. Notation of Hypercomplex Numbers**

The concept of hypercomplex numbers covers a broad range of numbers. In order to be most general we define a hypercomplex number \( z \in \mathbb{A}_N \) of order \( N \) as the sum of a real part and an \( N \) imaginary components.\(^1\)

\[
z = a_0 + \sum_{n=1}^{N} j_n a_n
\]

We have the basis 1 for the real part and the basis \( j_n \) for the \( n \)-th imaginary part. The properties of each basis \( j_n \) are defined by the respective hypercomplex algebra.

In the case of complex numbers it holds that \( j_1 = j \) and \( j^2 = -1 \). Additionally, quaternions are an important type of hypercomplex numbers. A quaternion \( q = a_0 + ia_1 + ja_2 + ka_3 \) has three imaginary units that are defined via \( i^2 = j^2 = k^2 = ij = k = -1 \) and \( i j = k \), \( jk = i \), \( k i = j \).

Alternatively, a hypercomplex number may be written as a vector \( \mathbf{z} \in \mathbb{R}^{(N+1) \times 1} \) by stacking all the components one below the other.

\[
\mathbf{z} := \begin{bmatrix} a_0 & a_1 & \ldots & a_N \end{bmatrix}^T \in \mathbb{R}^{(N+1) \times 1}
\]

The underline operation used above is commonly used for vectorized representations of (hyper-)complex numbers \([9]\). It may also be used for hypercomplex vectors. Thus, a hypercomplex vector \( \mathbf{z} = [z_1, \ldots, z_M] \) is converted into a corresponding real representation \( \mathbf{z'} \) using the underline operation as follows:

\[
\mathbf{z'} := \begin{bmatrix} \underline{\mathbf{z}} \ldots \underline{\mathbf{z}} \end{bmatrix}^T \in \mathbb{R}^{(N+1)M \times 1}
\]

**C. Hypercomplex Multiplication**

The multiplication of two hypercomplex numbers \( a \in \mathbb{A}_N \) and \( b \in \mathbb{A}_N \) is defined by the associated hypercomplex algebra. It may be described either by a table of products of imaginary units or via matrix-vector equations. We do not want to be restricted to commutative algebras. Thus, we define the two operators \( \cdot'_{L} \) and \( \cdot'_{R} \).

\[
(A \cdot'_{L} B)_{i,j} := \sum_{\ell=1}^{L} [A]_{i,\ell} [B]_{\ell,j}
\]

\[
(A \cdot'_{R} B)_{i,j} := \sum_{\ell=1}^{L} [B]_{i,\ell} [A]_{\ell,j}
\]

As a result, we have \( (A \cdot'_{L} B)^T = B^T \cdot'_{R} A^T \). Both operators equal the conventional multiplication in the case of commutative algebras. If none of these operators is given explicitly we use \( \cdot'_{L} \) by convention.

There is more than one way to have a matrix representation of hypercomplex numbers. We will stick to the following:

\[
a \cdot b = \begin{bmatrix} a^T T_0 b & \ldots & a^T T_N b \end{bmatrix}^T
\]

For a product of two hypercomplex numbers \( a \) and \( b \), equation (1) shows how to compute each resulting component. The respective algebra is represented via the real \( ((N + 1) \times (N + 1)) \)-matrices \( T_k \) with \( k \in \{0, \ldots, N\} \). An algebra is commutative with respect to multiplication iff \( T_k = T_k^T \). Additionally, an algebra is associative with respect to multiplication iff the following condition holds:

\[
\sum_{n=0}^{N} (T_n^T \otimes t_{k,n+1}) = \sum_{n=0}^{N} (t_{k,n+1} \otimes T_n),
\]

\[
\forall k = 1, \ldots, N + 1
\]

Here, \( t_{k,n} \) and \( t'_{k,n} \) are the \( n \)-th column vectors of \( T_k \) and \( T_k^T \), respectively. The proof of the associativity condition is skipped due to space limitations.

From (1) we obtain the notation for all product elements as follows:

\[
a \cdot b = (I_{N+1} \otimes a^T) T b = (I_{N+1} \otimes b^T) T'^T a
\]

The algebra is incorporated using the real \( ((N + 1)^2 \times (N + 1)) \)-matrices \( T \) or \( T' \), respectively.

\[
T = \begin{bmatrix} T_0 & \vdots & T_N \end{bmatrix}, \quad T' = \begin{bmatrix} T_0' & \vdots & T_N' \end{bmatrix}
\]

In the case of complex numbers the algebra is defined by the two multiplication matrices \( T_0 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( T_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

The quaternion algebra in matrix representation is given by:

\[
T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

**D. Hypercomplex Conjugation**

For our investigations we define an abstract conjugation function called \( \text{conj}_i(\cdot) \), with \( i \in \{1, \ldots, N\} \), that produces the \( i \)-th conjugate of a hypercomplex number. We presume that \( \text{conj}_i(\cdot) \) is an involution, i.e., \( \text{conj}_i(\text{conj}_i(z)) = z \) (see \([6]\)).

In this sense the first conjugate of a complex number \( z = a_0 + j a_1 \in \mathbb{C} \) is the ordinary complex conjugate, i.e., \( \text{conj}_1(z) := a_0 - ja_1 = z^* \).

An example of self-inverse mappings of quaternions is given in \([6]\). These can be used to define a possible first, second and third conjugate of a quaternion:

\[
\text{conj}_1(q) = -iu = a_0 + ia_1 - ja_2 - ka_3
\]

\[
\text{conj}_2(q) = -jy = a_0 - ia_1 + ja_2 - ka_3
\]

\[
\text{conj}_3(q) = -kq = a_0 - ia_1 - ja_2 + ka_3
\]

Additionally, we also define a zeros conjugate for notational convenience. It is always defined as the identity function \( \text{conj}_0(z) := \text{id}_{\mathbb{A}_N}(z) = z \).

In general there are multiple possible definitions of conjugates of a certain hypercomplex number. The actual choice is not important as long as all \( N + 1 \) conjugates together form
In other words, any hypercomplex number \( z \in \mathbb{A}_N \) can be expressed using another hypercomplex number \( z' \in \mathbb{A}_N \), with \( z' = 1 \alpha_0 + \sum_{n=1}^N j_n \alpha_n \), \( \alpha_n \neq 0 \):

\[
    z = \sum_{n=1}^N \alpha_n \mathrm{conj}_n(z'), \quad \alpha_n \in \mathbb{R}
\]

When using the real notation of hypercomplex numbers the \( \mathrm{conj}_1(.) \) operation turns into a matrix multiplication:

\[
    \mathrm{conj}_1(z) = C \overline{z}, \quad C_i \in \mathbb{R}^{(N+1)\times(N+1)}
\]

(2)

Since we only allow self-inverse conjugation functions, it must hold that \( C_i C_i = I_{(N+1)} \). Hence, \( C_i \) must be orthonormal and symmetric. From the above definition it follows that \( C_0 := I_{N+1} \). The functions \( \mathrm{conj}_1(.) \) or equivalently the matrices \( C_i \) serve as an alternative basis for the respective hypercomplex algebra.\(^2\) Thus, they must be mutually independent of each other.

The first conjugation in \( \mathbb{C} \) is \( C_1 = T_0 \). Additionally, the quaternion example given above results in \( C_1 = \mathrm{diag}(1 \ 1 \ 1 \ 1) \), \( C_2 = \mathrm{diag}(1 \ 1 \ 1 \ 1) \) and \( C_3 = \mathrm{diag}(1 \ 1 \ 1 \ 1) \).

In order to easily define a mean error, we also need to define an abstract overall conjugation \( z^* := \mathrm{conj}(z) \) of a hypercomplex number \( z \). It has to be chosen so that \( z^* \cdot z \) is real and hence introduces some concept of modulus. As for the real notation, we use the real \((N+1)\times(N+1)\)-matrix \( C \) so that \( \mathrm{conj}(z) = C \overline{z} \). The overall conjugate can be used to describe how signal power is measured for a certain type of hypercomplex numbers. Similar to the \( i \)-th conjugation the overall conjugation is defined as an involution as well. For the modulus criterion it must hold that the matrix products \( C T_i \) are skew-symmetric matrices \([1] \) for any imaginary part \((k = \{1, \ldots, N\})\).

Equivalently to the complex Hermitian transpose operation we will also use the notation \( z^H := (z^*)^T \).

A specialty of the complex numbers is, according to the above definitions, that the overall conjugation equals the first conjugate, i.e., \( C = C_1 = T_0 \). Similarly, one could choose \( C = T_0 \) in the quaternionic case.

### III. Data Model

Throughout this document a single-input multiple-output (SIMO) system is assumed. Therefore, in this section the notation for a hypercomplex SIMO model is derived for the hypercomplex domain and the real domain.

#### A. SIMO Model

The following single-input multiple-output (SIMO) system model is used:

\[
    y = h x + n \tag{3}
\]

The model presented in (3) is generic to any type of hypercomplex number \( \mathbb{A}_N \). Therefore, we have the output vector \( y \in \mathbb{A}_N^{M \times 1} \), the channel \( h \in \mathbb{A}_N^{M \times 1} \), the input \( x \in \mathbb{A}_N \) and the noise vector \( n \in \mathbb{A}_N^{M \times 1} \).

Additionally, a description of the input-output relation (3) in real notation is needed. To this end, we utilize the following notation:

\[
    y = \hat{H} T x + n \tag{4}
\]

The channel in real notation is given by

\[
    \hat{H} := \begin{bmatrix}
        I_{N+1} \otimes h_1 \\
        \vdots \\
        I_{N+1} \otimes h_M
    \end{bmatrix} \in \mathbb{R}^{(N+1)M \times (N+1)M}
\]

and the influence of the algebra is present via \( \hat{T} := I_M \otimes T \).

#### B. Widely Linear SIMO Model – Hypercomplex Notation

In complex widely linear filtering the input \( x \) is estimated via the output signal \( y \) and its complex conjugate \( y^* \) via

\[
    \hat{x} = w^H y + \sum_{n=1}^N w_n \mathrm{conj}_n(y) = w^H y_a \tag{5}
\]

In this notation \( w_a = [w_{n,1} \ldots w_{n,M}] \) is the weight vector for the \( n \)-th component. All weight vectors are stacked together into a single vector \( w \). The augmented vector \( y_a \) now contains \( y \) and all of its \( i \)-th conjugates \( y_a := \begin{bmatrix} y^T & \text{conj}_1(y) & \ldots & \text{conj}_N(y) \end{bmatrix}^T \). Both vectors, \( w \) and \( y_a \), are of size \( M(N+1) \times 1 \). In conjunction with (3) and (5) the estimated value of \( x \) is given by

\[
    \hat{x} = w^H H_a x_a + w^H n_a \tag{6}
\]

where

\[
    x_a := \begin{bmatrix}
        x^T & \text{conj}_1^T(x) & \ldots & \text{conj}_N^T(x) \\
    \end{bmatrix}^T \quad H_a := \begin{bmatrix}
        \text{conj}_1(h) \\
        \vdots \\
        \text{conj}_N(h)
    \end{bmatrix} \quad n_a := \begin{bmatrix}
        n^T & \text{conj}_1^T(n) & \ldots & \text{conj}_N^T(n) \\
    \end{bmatrix}^T
\]

are the augmented output vector, the augmented channel matrix and the noise vector, respectively.

#### C. Widely Linear SIMO Model – Real Notation

Starting from (5) and using the rules of multiplication (1) one can derive the following equation for the \( k \)-th component of the output vector:

\[
    \hat{x}_k = \sum_{m=1}^M \sum_{n=0}^N w_{n,m}^T C T_k y_n
\]
The matrix \( A_k \) is defined as follows:

\[
A_k := \begin{bmatrix}
I_M \otimes CT_k C_0 \\
\vdots \\
I_M \otimes CT_k C_N
\end{bmatrix}
\]

The estimated hypercomplex input \( \hat{x} \) in real notation can be expressed as follows:

\[
\hat{x} = (I_{N+1} \otimes \hat{H}^T \hat{x})^T A w + (I_{N+1} \otimes n)^T A w \tag{7}
\]

Equation (8) is the real description of the widely linear SIMO filtering vector given in (6). The task is to find the optimal solution.

\[
A := \begin{bmatrix}
A_0^T \\
\vdots \\
A_N^T
\end{bmatrix}
\]

The filter equation in real notation is obtained by plugging (4) into (7)

\[
\hat{x} = (I_{N+1} \otimes \hat{H}^T \hat{x})^T A w \tag{8}
\]

Equation (8) is the real description of the widely linear SIMO filtering vector given in (6). The task is to find the optimal filtering vector \( w \).

IV. DERIVATION OF WIDELY LINEAR FILTERS

Wide linear filters exploit the augmented second order statistics, meaning that correlations among the components of hypercomplex numbers are considered. We now show the derivation of the widely linear (WL) filter for the hypercomplex domain using the abstract concept of modulus. Afterwards, we present a minimum mean square error (MMSE) approach in the real domain which imposes less constraints on the algebra than the first proposed filter.

All following derivations presume that the input \( x \) and the noise \( n \) have zero mean. Moreover, input and noise are statistically independent.

A. Widely Linear Filter – Hypercomplex Notation

Wirtinger’s Calculus [2] is used to define a derivative of functions that map complex values to real values. In the hypercomplex domain we need the concept of modulus introduced above to obtain real-valued error functions. To this end, we propose a straightforward extension of Wirtinger’s Calculus. The derivatives are computed by treating \( w \) and \( w^H \) as independent variables.

The presented results are applicable to distributive and associative algebras. Under these conditions it is true that \( \text{tr}(A \odot B) = \text{tr}(B \odot A) \). Moreover, for the sake of simplicity we will assume that the overall conjugation of two hypercomplex numbers \( z_1, z_2 \in \mathbb{A}_N \) satisfies the condition \( z_1^* z_2^* = (z_1 z_2)^* \). This is a rather strict condition. The existence of such a conjugation operation is not necessarily ensured for all kinds of algebras. If such a conjugation is available, it follows that \( (A \odot B)^H = B^H \odot A^H \).

The above equation turns into the following form by utilizing the associativity, distributivity and trace operation:

\[
(H \odot \hat{R})_{xx} \hat{H}^H + R_{nn}(a) \hat{H}^T w = H \odot \hat{R}_{xx} (a)_{xx,0} \tag{9}
\]

The various variables introduced are the augmented input covariance matrix

\[
R_{xx}^{(a)} := \mathbb{E}^T \{ x^*_a x_a^T \} = \begin{bmatrix} x^*_{xx,0} & \cdots & x^*_{xx,N} \end{bmatrix}
\]

and the augmented noise covariance matrix \( R_{nn}^{(a)} := \mathbb{E}^T \{ n^*_a n_a^T \} \).

The solution to (9) with respect to \( w \) leads to the widely linear filter coefficients. In the special case where the modulus is equal to the squared sum of the hypercomplex components, i.e., \( z^* z = ||z||^2 \), this type of filter turns out to be a widely linear MMSE filter.

B. Widely Linear Filter – Real Notation

Solution (9) has the advantage that it can be computed in the respective hypercomplex domain. However, it has the drawback of presuming several properties of the hypercomplex algebra. Therefore, we propose a MMSE solution based on the real notation without any constraints on the algebra itself.

We start by computing the mean square error with respect to the real and imaginary parts of the estimated input \( \hat{x} \). Please note that this does not necessarily minimize the hypercomplex modulus as given in the last section.

\[
\frac{\partial}{\partial w} \mathbb{E} \{ \| \hat{x} - x \|^2 \} = 0
\]

The following equation is obtained after several manipulations:

\[
0 = A^T \left( I_{N+1} \otimes \hat{H}^T \hat{R}_{xx} \hat{H}^T \right) A w + A^T \left( I_{N+1} \otimes \hat{R}_{nn} \right) A w - A^T \mathbb{E} \{ \left( I_{N+1} \otimes (\hat{H}^T \hat{x}) \right) \hat{x} \} \tag{10}
\]

Similar to the equation in the hypercomplex domain we have the component-wise input covariance matrix \( \hat{R}_{xx} := \mathbb{E} \{ \hat{x} \hat{x}^T \} \) and the component-wise noise covariance matrix \( \hat{R}_{nn} := \mathbb{E} \{ n n^T \} \). The expectation operator in the last row of (10) can be simplified to this expression:

\[
\mathbb{E} \{ \left( I_{N+1} \otimes (\hat{H}^T \hat{x}) \right) \hat{x} \} = \text{vec} \left( \hat{H}^T \hat{R}_{xx} \right)
\]

Equation (10) can be solved with respect to \( w \) explicitly if the inverse of the square matrix \( A \) exists for a given algebra.

\[
w = A^{-1} \left[ I_{N+1} \otimes \left( \hat{H}^T \hat{R}_{xx} \hat{H}^T + \hat{R}_{nn} \right)^{-1} \right] \cdot \text{vec} \left( \hat{H}^T \hat{R}_{xx} \right)
\]

The last expression can be rewritten in a more compact form. Hence, we obtain the final result:

\[
w = A^{-1} \text{vec} \left( \left( \hat{H}^T \hat{R}_{xx} \hat{H}^T + \hat{R}_{nn} \right)^{-1} \hat{H}^T \hat{R}_{xx} \right) \tag{11}
\]
Equation (11) is the proposed abstract widely linear filter written in real notation. The only condition that has to be fulfilled is that $A^{-1}$ exists. Indeed, this is the case for most of the applied algebras.

The algebra’s properties are provided via the matrices $T$ and $A$. As a fortunate side effect (11) offers a more generic way of programming and testing. The hypercomplex number does not have to be implemented as a new type since it simply uses a vectorized real representation. Instead, just the matrices $T_k$ have to be replaced for any new type of hypercomplex number.

**C. Covariance Matrices**

In order to finish the theoretic analysis we now shed some light on the relationship between augmented covariance matrices and real component-wise covariance matrices.

An augmented SIMO covariance matrix of a random variable $z \in \mathbb{A}_N$ is defined as follows:

$$R_{zz}^{(a)} := \mathbb{E}^T \{ z_n^* z_n^T \} = \mathbb{E} \{ z_n^* \cdot z_n \}$$

Here, $z_n \in \mathbb{A}_N^{(N+1) \times 1}$ is the augmented vector of $z$. Now, let us express the $(i,j)$-th entry of the covariance matrix $R_{zz}$ using the real-valued notation:

$$[R_{zz}^{(a)}]_{i,j} = \begin{bmatrix} \mathbb{E} \{ (C)_{ij}^T T_0 C_{ij} \} \\ \vdots \\ \mathbb{E} \{ (C)_{ij}^T T_N C_{ij} \} \end{bmatrix}$$

This can equivalently be expressed as follows:

$$[R_{zz}^{(a)}]_{i,j} = \begin{bmatrix} \mathbb{E} \{ \text{tr} (C_j T_0 C_{ij} z_j^T) \} \\ \vdots \\ \mathbb{E} \{ \text{tr} (C_j T_N C_{ij} z_j^T) \} \end{bmatrix}$$

Under the condition that the expectation operator and the trace operator can be exchanged we obtain the desired relation between $R_{zz}^{(a)}$ and $\hat{R}_{zz}^{(a)}$:

$$[R_{zz}^{(a)}]_{i,j} = \begin{bmatrix} \text{tr} (C_j T_0 C_{ij} \hat{R}_{zz}) \\ \vdots \\ \text{tr} (C_j T_N C_{ij} \hat{R}_{zz}) \end{bmatrix}$$

From (12) it is clear that the augmented covariance matrix can be computed when the component-wise covariance matrix $\hat{R}_{zz} := \mathbb{E} \{ z z^T \}$ is known and vice versa.

**V. CONCLUSION**

Hypercomplex numbers gain more and more importance in different fields of science. However, advances in hypercomplex theory are mostly encountered for certain algebras. In this paper, we showed how to abstract from the concept and how to handle hypercomplex numbers in a much more holistic way.

Additionally, we presented two kinds of widely linear filters. Such filters incorporate the second order statistics of the input data as well. To this end, two different kinds of filters have been derived. The first widely linear filter can be computed directly in the hypercomplex domain. Though, its drawback is that it is restricted to associative and distributive algebras and that it needs a more stringent concept of conjugation.

In order to get rid of these restrictions, we derived a new kind of filter that is entirely computed in the real domain. The respective hypercomplex algebra is represented via special matrices specific to each algebra.

The shown concepts and abstractions are by no means exhausted. For instance, Wirtinger’s Calculus shows several more possibilities in this way, like derivatives with respect to each $i$-th conjugate.

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